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Conserved Quantities, Invariant Forms, and Dynamical Symmetries

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The relationships between conserved quantities and invariant forms are discussed in the framework of Lagrangian mechanics. As a consequence, it is shown that every dynamical symmetry is canonically related to a differential form which identifies a family of conserved quantities. Specific examples are also exhibited.

1. INTRODUCTION

The most commonly used procedures for the identification of constant of motion in Lagrangian mechanics depend on the connections between conserved quantities and the symmetry properties of the given system (see Sarlet et al., 1981a, for a comparative survey of most recent results).

In this paper we develop an alternative to the aforementioned mode of determining first integrals of motion, by relating them to the existence of suitably defined invariant forms. More precisely, we modify the proof of the Hojman–Harleston theorem recently given by Lutzky (1982) in order to show that one can always associate a family of conserved quantities with every differential form which is invariant along the trajectories of the given Lagrangian system in the extended tangent space. In so doing we also provide a constructive way for the explicit generation of the conserved quantities.

By making use of this general conclusion, we will also find a class of constants of motion determined by every dynamical symmetry Y via the introduction of an invariant 2-form canonically related to Y. After a

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comparative discussion of some implications of this result, a few specific examples are finally exhibited.

2. PRELIMINARIES

This section is devoted to a very brief review of some basic results and definitions. The reader is referred to Sarlet et al. (1981*a*) and to Crampin (1977) for a more exhaustive treatment of the subject.

Consider a configuration manifold M and denote by $\mathsf{R}\times\mathsf{TM}$ the associated extended tangent space, referred to local coordinates $(t, q^a, \dot{q}^a)(a = 1,\ldots,n)$. Suppose that a regular Lagrangian $L(t, q, \dot{q})$ is given. Then it may be shown that the solution of the normalized equations of motion

$$d\dot{q}^a/dt = \Lambda^a(t, q, \dot{q}) \tag{1}$$

is equivalent to the determination of the integral curves of the field Γ defined by

$$\Gamma = \partial/\partial t + \dot{q}^a\partial/\partial q^a + \Lambda^a\partial/\partial\dot{q}^a \tag{2}$$

Alternatively, Γ may be characterized as a solution of the equations

$$i_\Gamma d\theta = 0 \tag{3a}$$

$$i_\Gamma dt = 1 \tag{3b}$$

where the Cartan form θ, which is related to the given Lagrangian L by

$$\theta = (L - \dot{q}^a\partial L/\partial\dot{q}^a)\,dt + \partial L/\partial\dot{q}^a\,dq^a \tag{4}$$

is the pull back of the well-known fundamental form $p_a\,dq^a - H\,dt$ under the Legendre transformation.

3. CONSERVED QUANTITIES ASSOCIATED WITH INVARIANT FORMS

In the sequel we describe a constructive procedure leading to the determination of conserved quantities identified by differential forms with vanishing Lie derivative $\mathscr{L}$ along the flow of Γ. From the mathematical viewpoint, our approach is based on a modification of the proof of the Hojman–Harleston theorem recently proposed by Lutzky (Hojman et al., 1981; Lutzky, 1982).

Namely, we consider a family of $(2n+1)$-forms Ω_k ($k=0,\dots,s$) satisfying the condition

$$\mathscr{L}_\Gamma \Omega_k = 0 \tag{5}$$

Every Ω_k may be expressed as

$$\Omega_k = \rho_k \, dt \wedge dq^1 \wedge \cdots \wedge dq^n \wedge d\dot{q}^1 \wedge \cdots \wedge d\dot{q}^n \tag{6}$$

where ρ_k is a differentiable function. Then it may be shown that the ratio of any two coefficients ρ_k and ρ_j is a constant of motion. Actually, we have

$$0 = \mathscr{L}_\Gamma \Omega_k = \mathscr{L}_\Gamma(\rho_k \Omega_j/\rho_j) = \Gamma(\rho_k/\rho_j)\Omega_j \tag{7}$$

from which follows $\Gamma(\rho_k/\rho_j)=0$. Accordingly, we conclude that ρ_k/ρ_j is a conserved quantity.

We will now describe simple methods for the generation of invariant $(2n+1)$-forms. First, consider a $2p$-form ω and a $2q$-form λ satisfying

$$\mathscr{L}_\Gamma \omega = 0 \tag{8a}$$

$$\mathscr{L}_\Gamma \lambda = 0 \tag{8b}$$

In correspondence with every pair of integers k and r such that $pk+qr=n$, we may construct the $(2n+1)$ form

$$\Omega_k = dt \wedge (\wedge\, \omega)^k \wedge (\wedge\, \lambda)^r \tag{9}$$

for which equation (5) holds identically, in view of (8) and (3b).

Secondly, suppose that ω is a $(2p-1)$-form and fulfills (8a) but is not closed. Then we may recall the identity $\mathscr{L}_\Gamma d\omega = d\mathscr{L}_\Gamma \omega$ to conclude that replacing ω by $d\omega$ into the expression (9) we obtain an invariant $(2n+1)$-form. Of course, similar remarks also hold for the form λ.

Thirdly, it is to be noticed that the 2-form $d\theta$ does satisfy the requirement $\mathscr{L}_\Gamma d\theta = 0$, in view of (3a). Therefore, we may always associate conserved quantities to a single invariant form ω, provided the canonical choice $\lambda = d\theta$ has been made. In this case the expression of Ω_k is given by

$$\Omega_k = dt \wedge (\wedge\, \omega)^k \wedge (\wedge\, d\theta)^r \tag{10}$$

where $r = n - pk$ and k goes from 0 to the maximum integer not greater than n/p.

We conclude this section by the following remark. In a sense, the above results may be considered as a refinement of the connection between invariant differential forms and conserved quantities established by the following statement: every form ω satisfying the condition $\mathscr{L}_{f\Gamma}\omega = 0$, where f is any differentiable function, may be locally expressed in terms on $2n$ first integrals and of their differentials (Choquet Bruhat, 1968). The main advantage of the present formulation consists on the fact that it gives the detailed expression of the conserved quantities, without requiring any additional integration procedure, provided at least one invariant form is known.

4. CONSERVED QUANTITIES AND DYNAMICAL SYMMETRIES

In this section we introduce invariant 2-forms canonically related to the symmetry generators of the given Lagrangian system. We begin by recalling a few definitions.

A vector field Y with coordinate representation

$$Y = \tau(t, q, \dot{q})\,\partial/\partial t + K^a(t, q, \dot{q})\,\partial/\partial q^a + \eta^a(t, q, \dot{q})\,\partial/\partial \dot{q}^a \tag{11}$$

is said to be a *dynamical symmetry* (DS) iff

$$\mathscr{L}_Y\Gamma = [Y, \Gamma] = g\Gamma \tag{12}$$

where g is a differentiable function. It follows from (12) that DSs may be regarded as generators of infinitesimal transformations mapping integral curves of Γ into integral curves. If the components τ and K^a do not depend on $\dot{q}$ the vector Y is usually referred to as *point symmetry*. A *Noether symmetry* is characterized by the property

$$\mathscr{L}_Y d\theta = 0 \tag{13}$$

Every Noether symmetry is a DS (Sarlet et al., 1981a).

To the aim of establishing a correspondence between DSs and conserved quantities, let us turn our attention to the 1-form α defined by

$$\alpha = i_Y d\theta \tag{14}$$

that has already been introduced (Sarlet et al., 1981b) as a basic tool in the construction of the so-called higher-order Noether symmetries. Then the following useful identities can be proved by straightforward calculations, taking also into account equations (3) and (12):

$$i_\Gamma \alpha = 0 \tag{15a}$$

$$\mathscr{L}_\Gamma \alpha = 0 \tag{15b}$$

$$d\alpha = \mathscr{L}_Y d\theta \tag{15c}$$

$$i_\Gamma d\alpha = 0 \tag{15d}$$

If α is closed, equation (15c) implies that Y is a Noether symmetry, so that it gives rise to a Noether-type conserved quantity (Sarlet et al., 1981a; Crampin, 1977).

Moreover, it follows from equation (15d) that substitution of $d\theta$ by $d\theta + d\alpha$ into the definition (3) of Γ does not alter the vector field Γ. Looking at this property from a different viewpoint, we find that equations (15c) and (15d) imply $i_\Gamma d(\mathscr{L}_Y\theta) = 0$; if Y is a point symmetry, it follows by comparison with (3) that $\mathscr{L}_Y\theta$ may be regarded as a Cartan form associated with the Lagrangian $L' = Y(L) + \Gamma(\tau)L$, which gives rise to equivalent equations of motion and to a set of conserved quantities (Lutzky, 1979a, 1982).

Although in general an arbitrary DS Y does not identify such a Cartan form because $\mathscr{L}_Y\theta$ depends on the differentials $d\dot{q}^a$ and thus cannot assume the form (4), nevertheless we may always find a family of conserved quantities associated with Y. To this aim, we recall that the form α is invariant along the flow of Γ as a consequence of (15b). Of course, this implies that the 2-form $d\alpha$ is invariant too. Then, according to the discussion of the previous section, $d\alpha = \mathscr{L}_Y d\theta$ may be substituted into equation (10) giving rise to the conserved quantities ρ_k/ρ_j, where the coefficients k and j run from 0 to n.

As a first comment, it is to be remarked explicitly that when the DS Y degenerates into a point symmetry the first integrals ρ_k/ρ_j reduce to the ones already found by Lutzky (1982).

We also notice that the expression of the first integrals is simply obtained by algebraic manipulations of the partial derivatives of L and of the components of Y. From this viewpoint, our approach seems to be more efficient than those based on the determination of the so-called Noether-type conserved quantities (Sarlet et al., 1981a; Lutzky, 1979b) because they require the integration of suitably defined partial differential equations.

5. EXAMPLES

Aiming at an illustration of our approach to first integrals of motion, we consider the equations of motion of the one-dimensional harmonic oscillator modelled by the Lagrangian $L=\frac{1}{2}\dot{q}^2-\frac{1}{2}q^2$. The Cartan form θ is given by

$$\theta=-\tfrac{1}{2}(\dot{q}^2+q^2)\,dt+\dot{q}\,dq \tag{16}$$

and a DS Y may be written in the form (Lutzky, 1979b)

$$Y=q\dot{q}\partial/\partial t-q^3\partial/\partial q-(2q^2\dot{q}+\dot{q}^3)\partial/\partial\dot{q} \tag{17}$$

On using (10), (16), and (17) it is found that

$$\Omega_0=-\,dt\wedge dq\wedge d\dot{q} \tag{18}$$

$$\Omega_1=dt\wedge\mathscr{L}_Y d\theta=-2(\dot{q}^2+q^2)\,dt\wedge dq\wedge d\dot{q} \tag{19}$$

Finally, comparing (18) and (19) with (6), it follows that the first integral ρ_1/ρ_0 associated with the DS (17) is twice the well-known energy integral.

Suppose now that a Lagrangian $L=\frac{1}{2}g_{ab}(q)\dot{q}^a\dot{q}^b$ is given, where the metric form g_{ab} is either positive definite or of normal hyperbolic type. In the latter case, if $n=4$, the Lagrangian L models the geodesic motions of freely falling particles in the field of general relativity. It has already been shown that the Killing tensors of the metric g_{ab} may be regarded as generators of DSs for the geodesic equation, and families of conserved quantities associated with such DSs have also been exhibited (Caviglia, 1983a, b). A natural question then arises as to the possibility of identifying new first integrals of motion by the procedure described in this paper.

Actually, recalling the expression of the DS generated by a Killing tensor $K_{(a_1\ldots a_p)}$, namely,

$$Y=K^a_{a_2\ldots a_p}\dot{q}^{a_2}\ldots\dot{q}^{a_p}\partial/\partial q^a-\Gamma\left(K^a_{a_2\ldots a_p}\dot{q}^{a_2}\ldots\dot{q}^{a_p}\right)\partial/\partial\dot{q}^a \tag{20}$$

it may be verified by direct calculation that $d\alpha=\mathscr{L}_Y d\theta$ is a linear combination of 2-forms of the type $dq^a\wedge dq^b$, $dt\wedge dq^a$, and $dt\wedge d\dot{q}^a$. Then, after substitution into equation (10), it follows that $\rho_k=0$ for $k\neq 0$. Therefore, this example shows that a DS Y does not necessarily identify significant conserved quantities, even though $\mathscr{L}_Y d\theta$ is nonvanishing. Moreover, it may be remarked that the conserved quantities of the form ρ_k/ρ_j do not seem to be functionally related, *a priori*, to the Noether-type constant of motion which is known to be associated with the DS (20) (Caviglia, 1983b) because the latter constant does not vanish identically when Y is generated by a Killing tensor.

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